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Universality of symmetric one-dimensional random flights

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Abstract. The asymptotic behaviour of all the moments of a class of symmetric one-dimensional random flights, at the first few orders, are calculated. Different symmetric random flights are considered and it is shown that they simply differ up to a scaling in the number of jumps and that important statistical quantities are universal.

1. Introduction

It is well known that the random-walk model has relevant applications in many different fields. In particular, it is widely used for the study of transport (Bardeen and Herring 1952) and trapping processes (Rosenstock 1969, Montroll 1969) and well describes the behaviour of polymers and macromolecules (Yamakawa 1971, Flory 1969). It has also been applied to the analysis of micro-organism motion and, more generally, to fluctuation phenomena (Alt 1980, Levitt 1974). Some recent reviews (Weiss and Rubin 1983, Montroll and West 1979, Kher and Binder 1986) give wide outlooks on the matter and on the physical applications, and an accurate bibliography.

Several aspects of the problem are studied according to the processes one is interested in, but the span distribution and its moments are of general interest both from a mathematical point of view and for their physical relevance.

Different authors have derived, by using a Tauberian theorem (Hardy 1949), asymptotic expressions for the first two moments of the span distribution in the case of the standard one-dimensional random walk (Dvoretzky and Erdos 1951, Montroll 1964, Montroll and Weiss 1965).

An interesting generalisation of the random walk is the so-called random flight, in which the walker moves by jumping to any site with assigned probability. Blumen and Zumofen (1983) have investigated the incoherent energy transfer in ordered and disordered crystals by using a random flight model in which the jump probability distribution behaves as r^{-s} for multipolar interactions and as $\exp(-\gamma r)$ for exchange interactions (Forster 1949, Dexter 1953). The random flight has been related to other problems in statistical mechanics by Cummings and Stell (1983); they showed that the Ornstein-Zernike integral equations relative to the liquid and lattice-gas structure, to the percolation theory and to the random-flight problem are quite similar, so that many results in one of these areas can be extended to the others. The one-dimensional random flight applies, in particular, to the conductivity in linear and self-avoiding polymers (Chowdhury and Chakrabarti 1985). In this case, in fact, the charge can

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move not only to the nearest neighbour but also to a molecule which is occasionally close, the probability for this to happen being fixed only by the topology of the whole system.

The span distribution for a one-dimensional symmetric random flight was first given by Weiss and Rubin (1980). These authors, in 1976, derived the expressions for the first two moments of the spans in the case in which both the second and the fourth moments of the jump distribution exist. In the same paper they give the asymptotic behaviour of the span distribution when the second moment of the jump distribution is infinite.

In this paper we consider, in § 4, the case of a regular jump distribution. We derive the expressions for all the moments of the span distribution not only at the lowest but also at the next orders in the number N of steps. As mentioned above, the cases of the two first moments at the lowest order were already known. When the span moments are considered, the second-order corrections are $O(N^{-1/2})$, and not $O(N^{-1})$. It is interesting to notice that, in the case of standard random walk, the second-order corrections to the moments of the distribution of distinct sites visited are only $O(N^{-1})$. In § 5 the case of a jump distribution of the form $p(j) = \zeta(\alpha)^{-1} j^{-\alpha}$, $\alpha > 1$, is considered. We give the expressions for the span moments of any real power, to the first two orders in N, which are competitive for α close to 3.

We claim, however, that these results, although original, are not as relevant as the ones contained in § 6, where a general statement is given which relates the moments of the span distribution to the jump ones. Finally, the existence of universal (independent of $p(j)$) relations among the span moments is proved.

2. Definitions

In this paper we consider a one-dimensional symmetric random motion in which the steps of the walk have length j with probability $p(j)$ ($j \geqslant 1$). This motion is characterised by the structure function

$$\lambda(\phi) = \sum_{j=1} p(j) \cos j\phi \tag{1}$$

and by the moments of the steps (or jumps)

$$\langle j^\mu \rangle = \sum_{j=1} j^\mu p(j). \tag{2}$$

Weiss and Rubin (1980) have derived the exact expression for the probability $D(l, N)$ that, in a walk of N steps, the length l has been spanned

$$D(l, N) = d(l+2, N) - 2d(l+1, N) + d(l, N), \tag{3}$$

where

$$d(l, N) = 2f(l, N) - f(l/2, N) \tag{4}$$

and

$$f(l, N) \equiv \frac{1}{l} \sum_{k=1}^{[l/4]} \left(\lambda\left(\pi \frac{k}{l}\right)\right)^N \left(\cot^2 \pi \frac{k}{l} + (-1)^N \tan^2 \pi \frac{k}{l}\right). \tag{5}$$

In the following, the expression in the last set of brackets will be always approximated by $(2l/\pi k)^2 - \frac{1}{6}$, as the main contribution to $f(l, N)$ comes from those values of k which are much smaller than l and the fluctuations in N are asymptotically irrelevant.

3. The random flight distribution when the probability of large steps is exponentially decreasing

In this section we suppose that all the moments $\langle j^\mu \rangle$ are finite, as is the case when $p(j)$ is exponentially decreasing for large j. The structure function is then analytical and, as we are interested in small values of k, we can restrict the expansion of $\lambda(\phi)$ to fourth order in ϕ. With these approximations, $f(l, N)$ takes the form

$$f(l, N) = \frac{4l}{\pi^2} \sum_{k=1} \frac{1}{k^2} \exp\left(-\frac{\pi^2 N \langle j^2 \rangle}{2} \frac{k^2}{l^2}\right)\left[1 - \frac{\pi^2 k^2}{6 l^2}\left(1 + \frac{3\langle j^2 \rangle - \langle j^4 \rangle}{6} \frac{k^2}{l^2} N\right)\right]. \quad (6)$$

We now make use of the following exact consequence of the Poisson summation formula:

$$\sum_{h=1} \frac{\exp(-h^2 x)}{h^2} = \frac{\pi^2}{6} + (\pi x)^{1/2} - \frac{x}{2} + \sqrt{\pi} \sum_{h=1} \int_0^x \frac{\mathrm{d}t}{\sqrt{t}} \exp\left(-\frac{\pi^2 h^2}{t}\right) \quad (7)$$

and of the relations one obtains from (7) by derivation with respect to x. When these identities are used in (6) and only the most significant terms are maintained (the expression in square brackets in (6) is set to 1), the random flight distribution comes out to be well described by

$$D(l, N) = \left(\frac{32}{\pi N \langle j^2 \rangle}\right)^{1/2} \sum_{h=1} (-1)^h h^2 \exp\left(-\frac{h^2 l^2}{2N\langle j^2 \rangle}\right). \quad (8)$$

The probability that, after N steps, a length l has been spanned is then, a factor $(N\langle j^2 \rangle)^{1/2}$ apart, a function of $l/(N\langle j^2 \rangle)$, which is shown in figure 1.

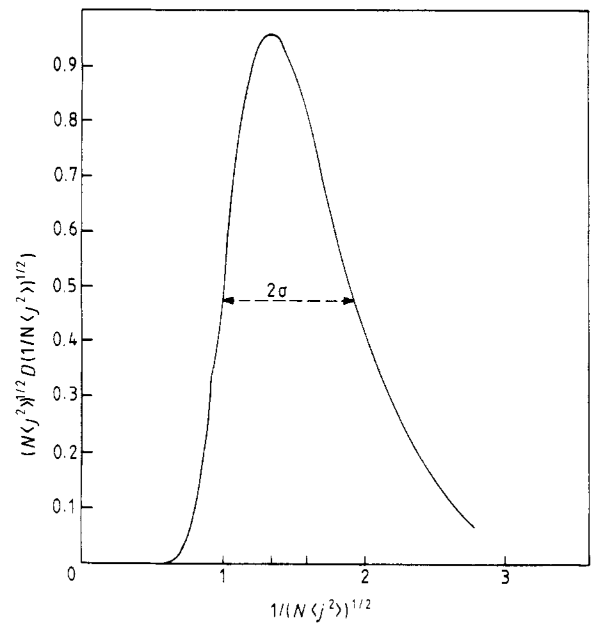

Figure 1. Asymptotic behaviour of the span distribution for very large l and N ($l/N^{1/2}$ being finite) and for a regular jump distribution.

If, in (8), we put $y=\exp(-l^2/(2N\langle j^2\rangle))$, it is easily seen that the distribution has its maximum for the value ($\neq 0$) which satisfies the equation

$$y=y\left(1-\frac{1-2^4y^3+3^4y^8-4^4y^{15}+\ldots}{1-2^6y^3+3^6y^8-4^6y^{15}+\ldots}\right), \qquad y<1. \tag{9}$$

This value is easily evaluated by iteration of (8) starting from the initial value $y=2^{-4/3}$. After two iterations it turns out that the maximum of the random motion distribution is at

$$l=1.345\,773\,458\ldots(N\langle j^2\rangle)^{1/2}.$$

We conclude this section by noting that, if all the moments of the distribution $p(j)$ of the steps are finite, the distribution of the spans is asymptotically dependent only on $\langle j^2\rangle$. This fact will have significant consequences. The comparison with the classical random walk is straightforward when one remembers that, in this case, all the moments $\langle j^\mu\rangle$ are equal to 1.

4. Span distribution moments when the probability of large steps is exponentially decreasing

It turns out to be convenient to derive the expressions for the moments of the spans $\langle l^\mu\rangle$ from the moments $\langle (l+1)^\mu\rangle$. For these moments we have

$$\begin{aligned}\langle (l+1)^\mu\rangle &= \sum_{l=1}^{\infty}(l+1)D(l,N)\\ &= \lim_{M\to\infty}\Bigg[(M+1)^\mu d(M+2,N)-(M+2)^\mu d(M+1,N)\\ &\quad +2\sum_{l=1}^{M+1}\sum_{\nu=1}^{\mu/2}\binom{\mu}{2\nu}l^{\mu-2\nu}d(l,N)\Bigg].\end{aligned} \tag{10}$$

The evaluation of the last term of (10) is now carried out through a systematic use, in (6), of the identity (7) and of its derivatives. These calculations give us first of all that the asymptotic behaviour of $d(l,N)$ for large l ($l\gg N$) is

$$d(l,N)\simeq l-\left(\frac{8N\langle j^2\rangle}{\pi}\right)^{1/2}-\left(\frac{2}{9\pi N\langle j^2\rangle}\right)^{1/2}\left(1+\frac{1}{4\langle j^2\rangle}(3\langle j^2\rangle^2-\langle j^4\rangle)\right). \tag{11}$$

Moreover, the double sums over l and h are calculated through the relation

$$\begin{aligned}&\sum_{l=1}^{M}\sum_{h=1} l^\mu\exp(-h^2l^2x)\\ &\qquad\simeq\frac{\zeta(\mu+1)}{2}\Gamma\left(\frac{\mu+1}{2}\right)x^{-(\mu+1)/2}+\left(\frac{\pi}{x}\right)^{1/2}\frac{\mu^2-\mu-6}{12\mu 2^\mu}-\frac{\mu^2+\mu-6}{6(\mu+1)2^{\mu+2}}\end{aligned} \tag{12}$$

(ζ being the Riemann zeta function), which is derived from the identity

$$\begin{aligned}&\int_{1/2} \mathrm{d}l\, l^\mu\exp(-h^2l^2x)\\ &\qquad=\sum_{l=1} l^\mu\exp(-h^2l^2x)+\sum_{j=1}\int_{j-1/2}^{j+1/2}\mathrm{d}l\,(l^\mu\exp(-h^2l^2x)-j^\mu\exp(-h^2l^2x)),\end{aligned} \tag{13}$$

where the last integral on the RHS is evaluated using the mean value theorem to the lowest significant order.

We obtain finally the following expressions for the moments

$$\langle (l+1)\rangle_N \simeq \left(\frac{8N\langle j^2\rangle}{\pi}\right)^{1/2}\left[1+\frac{1}{6N\langle j^2\rangle}\left(1+\frac{3\langle j^2\rangle^2-\langle j^4\rangle}{4\langle j^2\rangle^2}\right)\right], \tag{14a}$$

$$\langle (l+1)^2\rangle_N \simeq 4\ln 2N\langle j^2\rangle+\tfrac{4}{3}\ln 2+\tfrac{1}{6}, \tag{14b}$$

(actually the correction $\frac{1}{6}$ has been obtained by numerical calculations),

$$\langle (l+1)^3\rangle_N \simeq \tfrac{1}{3}(2\pi N\langle j^2\rangle)^{3/2}+\left(\frac{2N\langle j^2\rangle}{\pi}\right)^{1/2}\left[1+\frac{\pi^2}{3}\left(1-\frac{3\langle j^2\rangle^2-\langle j^4\rangle}{4\langle j^2\rangle^2}\right)\right], \tag{14c}$$

$$\langle (l+1)^\mu\rangle_N \simeq A(\mu)(N\langle j^2\rangle)^{\mu/2}-\left(B(\mu)-C(\mu)\frac{3\langle j^2\rangle^2-\langle j^4\rangle}{\langle j^2\rangle^2}\right)(N\langle j^2\rangle)^{-1+\mu/2}, \tag{14d}$$

where $\mu\geqslant 4$ and

$$A(\mu)=\frac{16}{\sqrt{\pi}}\frac{(2^{\mu-2}-1)}{2^{\mu/2}}\zeta(\mu-1)\Gamma\left(\frac{\mu+1}{2}\right), \tag{14e}$$

$$B(\mu)=\frac{\mu}{3}\left(\frac{(2^{\mu-4}-1)\zeta(\mu-3)}{(2^{\mu-2}-1)\zeta(\mu-1)}+\frac{1}{2}\right)A(\mu), \tag{14f}$$

$$C(\mu)=\tfrac{1}{24}\mu(\mu-2)A(\mu). \tag{14g}$$

In (14*f*) the term $(2^{\mu-4}-1)\zeta(\mu-3)$ must be substituted by $\ln 2$ if $\mu=4$. These results will be discussed in the last section.

In table 1 we show how good the results (14) are, in the particular case of the classical random walk.

Table 1. The first few moments of the number of distinct sites visited in a classical one-dimensional random walk at the first two orders in N.

N		$\langle (l+1)\rangle_N$	$\langle (l+1)^2\rangle_N$	$\langle (l+1)^3\rangle_N$	$\langle (l+1)^4\rangle_N$	$\langle (l+1)^5\rangle_N$
	First order	$(8N/\pi)^{1/2}$	$4N\ln 2$	$\frac{1}{3}(2\pi N)^{3/2}$	$9\zeta(3)N^2$	$\frac{7}{90}\pi(2\pi N)^{5/2}$
	Second order	$(2\pi N)^{-1/2}$	$\frac{4}{3}\ln 2+\frac{1}{6}$	$\frac{1}{12}(\pi^2+6)(8N/\pi)^{1/2}$	$4N\ln 2$	$\frac{1}{108}(60-7\pi^2)(2\pi N)^{3/2}$
100	1st	15.957 69	277.258 872	5249.869 98	108 185.2	2417 993
	2nd	15.997 58	278.349 735	5270.973 33	108 462.4	2416 668
	numerical	15.997 53	278.349 735	5271.003 71	108 463.7	2416 728
250	1st	25.231 32	693.147 18	20 751.933	676 157.0	23 894 894
	2nd	25.256 56	694.238 04	20 785.301	676 850.1	23 889 655
	numerical	25.256 54	694.238 043	20 785.320	676 851.5	23 889 751
500	1st	35.682 48	1386.294 361	58 695.331	2704 628.0	135 169 929
	2nd	35.700 32	1387.385 224	58 742.520	2706 014.3	135 155 113
	numerical	35.700 32	1387.385 224	58 742.535	2706 015.8	135 155 258
1000	1st	50.462 65	2772.588 722	166 015.47	10 818 512.0	764 636 587
	2nd	50.475 27	2773.679 585	166 082.21	10 821 284.6	764 594 680
	numerical	50.475 26	2773.679 585	166 082.20	10 821 286.6	764 595 033
2000	1st	71.364 96	5545.177 444	469 562.65	43 274 047.8	4325 437 723
	2nd	71.373 88	5546.268 307	469 657.02	43 279 593.0	4325 319 191
	numerical	71.373 88	5546.268 307	469 657.04	43 279 595.0	4325 320 173

5. The distribution $p(j)=j^{-\alpha}/\zeta(\alpha)$

The jump probability distribution $p(j)=j^{-\alpha}/\zeta(\alpha)$ is an excellent representative of those distributions which have no moments $\langle j^{\mu}\rangle$ for μ greater than $\alpha-1$. In fact the asymptotic behaviour of $p(j)$ for large j will dominate the properties of the span distribution $D(l, N)$ for large N. In the following we shall consider the different situations which happen for increasing α.

(a) $1<\alpha\leqslant 2$. In this case neither $p(j)$ nor $D(l, N)$ have any integer moment.

(b) $2<\alpha<3$. In the range $2<\alpha<4$ Hughes *et al* (1981) approximated the structure function for small values of its argument. In the following we need the better approximation

$$\lambda\left(\pi\frac{k}{l}\right)\simeq 1-\frac{\pi^2k^2}{2l^2}\frac{\zeta(\alpha-2)}{\zeta(\alpha)}+\frac{(k/l)^{1-\alpha}}{2^{\alpha}\zeta(1-\alpha)},\tag{15}$$

which is derived by applying the Poisson summation formula (Davis *et al* 1975) to

$$\sum_{j=1}\frac{1}{j^{\alpha}}(\cos j\phi-1+\phi^2/2),\qquad 3<\alpha<4,$$

and by analytically continuing the result to the region $2<\alpha\leqslant 3$ (notice that the RHS of (15) is analytical at $\alpha=3$).

The last term of (15) is dominant and we can thus write

$$\lambda^{N}\left(\pi\frac{k}{l}\right)\simeq\exp\left[\frac{N}{2^{\alpha}\zeta(1-\alpha)}\left(\frac{k}{l}\right)^{\alpha-1}\right]\left(1-\frac{\pi^2}{2}\frac{\zeta(\alpha-2)}{\zeta(\alpha)}\frac{k^2}{l^2}N\right).\tag{16}$$

In this case only the mean span is finite and only the first two terms on the RHS of (10) contribute to its value. If we use the approximation

$$\sum_{k=1}k^{\varepsilon}\exp(-\beta k^{\alpha-1})\simeq\frac{1}{\alpha-1}\Gamma\left(\frac{\epsilon+1}{\alpha-1}\right)\beta^{(\varepsilon+1)/(\alpha-1)},\qquad \varepsilon=0,2;\ \alpha>1;\ \beta\to 0,\tag{17}$$

we obtain

$$\begin{aligned}\langle(l+1)\rangle_N\simeq&\frac{2}{\pi^2}\frac{\alpha-1}{\alpha-2}\Gamma\left(\frac{2\alpha-3}{\alpha-1}\right)\left(-\frac{N}{2^{\alpha}\zeta(1-\alpha)}\right)^{1/(\alpha-1)}\\&+\frac{\zeta(\alpha-2)}{\zeta(\alpha)}\left(\frac{-1}{2^{\alpha}\zeta(1-\alpha)}\right)^{-1/(\alpha-1)}\Gamma\left(\frac{1}{\alpha-1}\right)N^{(\alpha-2)/(\alpha-1)},\end{aligned}\tag{18}$$

$(2<\alpha<3)$. Notice that, if α is close to 3, the two terms on the RHS of (18) are comparable and diverge for $\alpha\to 3$ in the same way.

(c) $\alpha=3$. The last consideration implies that, in this case, the last two terms of (15) must be treated on the same footing. The expression for the first moment is achieved through the following steps:

(i) $$\lambda^{N}\left(\pi\frac{k}{l}\right)\simeq\exp\left(-\frac{k^2}{l^2}N\gamma(\alpha)\right)\sum_{n=0}\frac{1}{n!}\left(\frac{N}{2^{\alpha}\zeta(1-\alpha)}\right)^{n}\left[\left(\frac{k}{l}\right)^{\alpha-1}-\left(\frac{k}{l}\right)^{2}\right]^{n},\tag{19}$$

where

$$\gamma(\alpha)=\frac{\pi^2}{2}\frac{\zeta(\alpha-2)}{\zeta(\alpha)}-\frac{1}{2^{\alpha}\zeta(1-\alpha)},$$

(ii) each term in $\lambda^{N}(\pi k/l)$ of the form $(k/l)^{\beta}\exp(-\gamma(\alpha)Nk^2/l^2)$ gives to $\langle(l+1)\rangle_N$ as main contribution $-(2/\pi^2)\Gamma((\beta-1)/2)(N\gamma(\alpha))^{(1-\beta)/2}$; this fact can be achieved, using (17), as described in § 1;

(iii) for α very close to 3, the generic term in the expansion (19) gives to $\langle (l+1)\rangle_N$ the contribution:

$$-\frac{2}{\pi^2}\frac{1}{n!}\left(\frac{N}{2^\alpha\zeta(1-\alpha)}\right)^n(\alpha-3)^n\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\Gamma\left(\frac{2n-1}{2}\right)(N\gamma(\alpha))^{(2n-1+z)/2}\right)\right]_{z=0}, \tag{20}$$

(iv) if we take the largest term (in N) and put $\alpha=3$, we obtain finally

$$\langle (l+1)\rangle_N \simeq \frac{2N^{1/2}\ln N}{[\pi\zeta(3)(3-2\ln\pi)]^{1/2}} \simeq 1.220\,95\ldots N^{1/2}\ln N. \tag{21}$$

(d) $3<\alpha<4$. In this case the first two moments of the jump probability distribution exist and so it should be for the first two moments of the span distribution. Since the second term on the RHS of (15) is now dominant we can write

$$\lambda^N\left(\frac{\pi k}{l}\right)\simeq \exp\left(-\frac{\pi^2}{2}\frac{\zeta(\alpha-2)}{\zeta(\alpha)}\frac{k^2}{l^2}N\right)\left[1+\frac{N}{2^\alpha\zeta(1-\alpha)}\left(\frac{k}{l}\right)^{\alpha-1}\right]. \tag{22}$$

The first moment is given by

$$\langle (l+1)\rangle_N \simeq 4\left(\frac{\zeta(\alpha-2)}{2\pi\zeta(\alpha)}\right)^{1/2}N^{1/2}-\frac{2^{1-\alpha}}{\zeta(1-\alpha)}\frac{1}{\pi^2}\left(\frac{\pi^2}{2}\frac{\zeta(\alpha-2)}{\zeta(\alpha)}\right)^{1-\alpha/2}\Gamma\left(\frac{\alpha-2}{2}\right)N^{(4-\alpha)/2}. \tag{23}$$

The derivation of the higher moments is based on the approximation

$$\sum_{l=1}^{M} l^{-\alpha}\sum_{k=1} k^\beta \exp(-\lambda k^2/l^2)\simeq \frac{\Gamma(\beta+\frac{1}{2})}{2\lambda^{\beta+1/2}}\sum_{l=1}^{M} l^{\beta-\alpha+1} + \frac{(\beta-\alpha+2)\Gamma((\beta-\alpha+2)/2)\Gamma((\alpha-1)/2)}{\pi^{\beta-\alpha+3/2}\Gamma((\alpha-\beta-1)/2)\lambda^{(\alpha-1)/2}}, \tag{24}$$

which holds for $\beta+2>\alpha>1$, while, for $\alpha\leqslant 1$, the last term in (24) diverges for $M\to\infty$.

Using (10), (4), (5), (22) and (24) we find, for $\mu\geqslant 3$,

$$\langle (l+1)^\mu\rangle_N = 16(2^{\mu-2}-1)\zeta(\mu-1)N^{\mu/2}\Gamma\left(\frac{\mu+1}{2}\right)\left(\frac{\zeta(\alpha-2)}{2\pi\zeta(\alpha)}\right)^{\mu/2}(1+\beta(\alpha,\mu)N^{(\alpha-3)/2}), \tag{25}$$

where

$$\begin{aligned}\beta(\alpha,\mu)&=\frac{(\mu-1)}{\pi^\alpha}\frac{\Gamma((\alpha-\mu-1)/2)(\zeta(\alpha-2)/\zeta(\alpha))^{(\mu-\alpha+1)/2}}{2^{\mu+\alpha/2}\zeta(1-\alpha)\Gamma(1-\mu/2)}, &&\text{if } \alpha>\mu+1\\ \beta(\alpha,\mu)&=\infty &&\text{if } \alpha\leqslant\mu+1\end{aligned} \tag{26}$$

and $(2^{\mu-2}-1)\zeta(\mu-1)$ must be substituted by $\ln 2$ if $\mu=2$.

Equations (25) and (24) imply that the second-order correction, in the asymptotic expression of the moment $\langle (l+1)^\mu\rangle_N$, may overcome the dominant term. This is the reason why we have systematically considered two orders in the expansions of the moments.

On the other hand, $\alpha>\mu+1$ is the condition for the existence of $\langle j^\mu\rangle$, so that we may state that

'$\langle l^\mu\rangle_N$ exists if and only if $\langle j^\mu\rangle$ exists'.

(e) $4\leqslant\alpha<6$. The previous result, which has a simple physical interpretation, has been derived when $3<\alpha<4$ and allowing μ to be continuous, but it will be true in

Table 2. Asymptotic behaviour of the first two moments for a random flight described by the distribution $p(j)=(j^{\alpha}\zeta(\alpha))^{-1}$ for some α.

α	$\langle(l+1)\rangle_N$	$\langle(l+1)^2\rangle_N = 4\ln 2\dfrac{\zeta(\alpha-2)N}{\zeta(\alpha)}$
2.1	$6.5299\ N^{0.9091}-0.1338\ N^{0.0909}$	∞
2.5	$1.9742\ N^{0.6667}-0.4054\ N^{0.3333}$	∞
2.9	$2.6237\ N^{0.5263}-1.8771\ N^{0.4737}$	∞
3.0	$1.220\,95\ N^{1/2}\ln N$	∞
3.1	$4.7731\ N^{0.5}-1.726\ N^{0.45}$	$24.805\ N$
3.5	$2.4294\ N^{0.5}-0.4144\ N^{0.25}$	$6.425\ N$
4.0	$1.9673\ N^{0.5}-0.2027$	$4.213\ N$
4.5	$1.7994\ N^{0.5}-0.1806\ N^{-0.25}$	$3.525\ N$
4.9	$1.7305\ N^{0.5}-0.6357\ N^{-0.45}$	$3.261\ N$
5.1	$1.7070\ N^{0.5}-0.0198\ N^{-0.5}$	$3.175\ N+1.0908$
6.5	$1.6280\ N^{0.5}+0.2289\ N^{-0.5}$	$2.890\ N+1.0908$
∞	$1.5958\ N^{0.5}+0.3989\ N^{-0.5}$	$2.773\ N+1.0908$

general. In the latter case the term $(1/4!)(\pi k/l)^4\ \zeta(\alpha-4)/\zeta(\alpha)$ must be added to the RHS of (15) to obtain a good approximation of $\lambda(\pi k/l)$. A simple generalisation of the result (25) implies that $\langle(l+1)^{\mu}\rangle_N$ is obtained by adding to the RHS of (25) a term which behaves like $N^{(\mu-2)/2}$. Once more we arrive at the last conclusion of the previous case.

(f) $\alpha \geqslant 6$. In this case only the moments $\langle l^{\mu}\rangle$ for $\mu<5$ exist and they are given by equations (14).

In the previous discussion we considered only the distribution $p(j)=\zeta(\alpha)^{-1}j^{-\alpha}$ but it is clear that only the asymptotic behaviour of $p(j)$ is relevant to determine the existence of the span moments. In particular, if a value of α exists such that

$$\lim_{j\to\infty} p(j)j^{\beta} = \begin{cases} \infty & \text{for any } \beta<\alpha \\ 0 & \text{for any } \beta>\alpha \end{cases}$$

then $\langle l^{\mu}\rangle$ exists if and only if $\mu<\alpha$.

The results expressed by (18), (21), (23) and (25) (for $\mu=2$) are numerically presented in table 2.

6. Conclusions

The moments of the one-dimensional random-flight distribution have been derived for a variety of probabilities $p(j)$ of jumping to sites at distance j. All these probabilities have been supposed to be symmetric and we feel that the results we have obtained in this paper will be drastically changed when asymmetric motions are considered. We have treated in § 4 the case of localised motions, in which the probabilities of large jumps are exponentially decreasing. Equation (14d) shows that, while the second-order correction in $\langle(l+1)^{\mu}\rangle_N$ is of order $(N\langle j^2\rangle)^{(-1+\mu)/2}$ with a coefficient depending on distribution $p(j)$, the first-order term simply scales the number of steps by the factor $\langle j^2\rangle$. This holds for any moment. It is interesting to notice that, if we consider the moments of the spans $\langle l^{\mu}\rangle_N$, not only the dominant term in $N^{\mu/2}$, but also the next one in $N^{(\mu-1)/2}$ depend on the jump probabilities $p(j)$ simply via the same scaling factor on N. Equation (25) shows that this conclusion holds also for the 'singular' jump distribution of the type $(\zeta(\alpha)j^{\alpha})^{-1}$ if $\alpha \leq 4$.

In general we may summarise in the following way.

'$\langle l^{\mu}\rangle_N$ exists if and only if $\langle j^{\mu}\rangle$ exists; moreover, if $\langle j^2\rangle$ and $\langle j^3\rangle$ exist, $\langle l^{\mu}\rangle$, if finite, scales simply the number of steps by the factor $\langle j^2\rangle$ both in the dominant term in $N^{\mu/2}$ and in the next one in $N^{(\mu-1)/2}$. The last one is equal to $\mu\langle l^{\mu-1}\rangle_N$. If $\langle j^2\rangle$ exists and $\langle j^3\rangle=\infty$, then $\langle l^{\mu}\rangle_N$ exists for any $\mu \leq 2$ and scales the number of steps by the factor $\langle j^2\rangle$ only in the dominant term. If $\langle j\rangle$ exists and $\langle j^2\rangle=\infty$, then $\langle l\rangle_N$ exists but there is no scaling; if $\langle j^{\mu}\rangle<\infty$ for $\mu<\alpha-1<2$, then $\langle l\rangle_N \simeq N^{1/(\alpha-1)}$ with an increased effective dimensionality.'

This last result agrees with the result of Hughes *et al* (1981) who, through an analysis of the asymptotic behaviour of the expected number of distinct sites visited found by Gillis and Weiss (1970), observed that the effective dimensionality of the walk increases in the range $1<\alpha<3$. As a matter of fact, we have derived our conclusions for the distributions $p(j)$ which behave asymptotically as $j^{-\alpha}$, but it is easily seen that they may be extended to a much larger class of distributions and we believe they hold in general.

An immediate consequence of the previous theorem is that the correlation effects for different symmetric random motions are equal up to a scaling. Next, 'for a symmetrical random flight, the ratio $\langle l^{\mu}\rangle/\langle l\rangle^{\mu}$ is asymptotically independent of the distributions of the jumps'. The fractional standard deviation, the kurtosis and the skewness (Kendall and Stuart 1958), when they exist, are universal constants (i.e. they are independent of the type of symmetric random walk), respectively equal to 0.2980 . . . , 0.9656 . . . and 4.382 . . . . The analytic expressions for these quantities may be derived from those given in table 1. These facts, however, do not imply that the form of the normalised distribution $D(l, N)$ for the spans, as given in figure 1, is universal. This is true when the structure function of the jumps is analytical but, if we are in one of the cases treated in § 5, $D(l, N)$ will be the sum of some exponentials (of the type of equation (8)) and of some other terms which behave like $l^{-\alpha}$ (for some α). As a consequence, while the distribution is practically unchanged in the central region $l \simeq (N\langle j^2\rangle)^{1/2}$, it will be modified for very large l in such a way that, although it does not appear in a plot, the moments of the spans will become infinite as soon as $\mu>\alpha-1$.

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